THE KCM ANNUAL CONFERENCE 2015

ENGAGE!

THE HILTON HOTEL

LEXINGTON, KY

MARCH 9-10, 2015

TITLE OF PRESENTATION: ENGAGE YOUR STUDENTS IN RICH PROBLEM SOLVING TASK CORRELATING TO THE CCCS

DATE AND TIME: MONDAY, MARCH 9, 2015, 9:25 A.M.-10:35 A.M.

LOCATION: CRIMSON CLOVER

PRESENTER: JAY L. SCHIFFMAN

ROWAN UNIVERSITY

ENGAGE YOUR STUDENTS IN RICH PROBLEM SOLVING TASKS CORRELATING TO THE CCCS

JAY L. SCHIFFMAN

__Abstract:__ **Good problem solving is paramount in our Common Core environment. This workshop will engage participants in rich problem tasks selected from algebra, geometry, number and operations, calculus and discrete mathematics. Participants working in small groups will reason, explore connections among disciplines with these rich tasks and apply appropriate tools strategically. Title: Engage Your Students in Rich Problem Solving Tasks correlating to the CCCS.**

SOME PROBLEMS AND DISCUSSION ACTIVITIES ON RICH PROBLEM SOLVING TASKS RELATED TO THE CCCS:

I. A Fun Activity with the Fibonacci sequence.
Consider the sum of any six consecutive terms in the Fibonacci sequence. Form the sum and divide by four. Try this with three different numerical data sets. Form a conjecture. Can you prove your conjecture? Repeat this problem for the sum of ten consecutive terms in the Fibonacci sequence. Form the sum and divide by eleven. Next consider the sum of any fourteen consecutive terms in the Fibonacci sequence. Form the sum and divide by twenty-nine.

II. Determine what occurs when one adds and multiplies the various combinations of even and odd integers. Form conjectures and create a table. Then try to justify formally.

III. Consider a rectangle with perimeters 20 and 32 units respectively. Determine the areas of all such rectangles having integer lengths and secure the dimensions of the rectangles with the largest possible areas. What do you conclude? Provide the necessary constraints on the dimensions of the rectangles. Can you generalize?

IV. Consider the sum of two consecutive unit fractions with even denominators such as $\frac{1}{2}+\frac{1}{4}$**. Do this for the first fifteen iterations. (i.e. Consider** $\frac{1}{4}+\frac{1}{6}, \frac{1}{6}+\frac{1}{8},$ **etc.) What do you notice when considering the numerators and denominators in each of these sums? Repeat this with the sum of two consecutive unit fractions with odd denominators such as** $\frac{1}{3}+\frac{1}{5}$**. Repeat for fifteen iterations. What do you notice about the numerators and denominators in the sums? Do you see any connections to geometry?**

V. Consider the entries in each row of Pascal's Triangle with regards to their parity (whether an entry is even or odd). Extend to thirty two rows of the triangle. Count the number of even and odd entries in each row. Form a table giving the cumulative total of the even entries, the odd entries, and the total number of entries. Are there any rows which have all odd entries? Are there any rows which have all even entries save the first and last 1's? Determine the ratio of the cumulative number of even entries to the total number of entries in the triangle. Form conjectures for all of the above. Use the calculator to determine the entries in the first five rows of the triangle as well as both the sum and cumulative sum of the entries in the initial five rows. Look at the entries on the first, second, third, fourth and fifth diagonals in the triangle. Find polynomial functions that fit the data. Generalize. Use the calculator to determine the entries in the first five rows of the triangle as well as both the sum and cumulative sum of the entries in the initial five rows.

VI. Consider the calendar for the month of MARCH 2015 below:

MARCH 2015

S	M	T	W	R	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28

a. Select several 3×3 groups of numbers and find the sum of these numbers. Determine how the obtained sums related to the middle number.

b. Prove that the sum of any 9 integers in any 3×3 set of numbers selected from a monthly calendar will always be equal to 9 times the middle number. Use algebra and technology to furnish a convincing argument.

VII. Determine if each of the following statements is true or false. If it is true, construct a formal proof. For all false statements, determine an appropriate counterexample and explain what needs to be altered to make the false statements true.

(a). The product of any three integers is divisible by 2.

(b). The sum of four consecutive integers is divisible by 4.

(c). The product of four consecutive integers plus one is always a perfect square.

(d). The sum of two prime numbers is never a prime number.

(e). The sum of five consecutive integers is divisible by 5.

(f). If the sides of a right triangle are tripled, then both the perimeter and area of the right triangle are tripled.

SOLUTIONS TO SOME PROBLEMS AND DISCUSSION ACTIVITIES ON RICH PROBLEM SOLVING TASKS RELATED TO THE CCCS

I. A Fun Activity with the Fibonacci sequence

We generate the Fibonacci sequence on the HOME SCREEN. First recall the famous *Fibonacci sequence* is recursively defined as follows:

Define $F_1 = F_2 = 1$ *and* $F_n = F_{n-2} + F_{n-1}$ *for* $n \geq 3.$ Here $F_n =$ the *nth* term of the Fibonacci sequence. We use the VOYAGE 200 to generate the initial forty outputs in the Fibonacci sequence. See **FIGURES 1-7:**

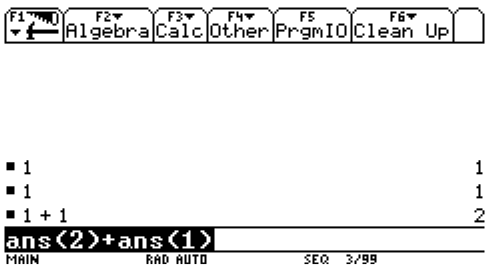

FIGURE 1

In **FIGURE 1,** on The HOME SCREEN, we entered the initial two terms to start the recursion which are both 1 and then used the command $ans(2) + ans(1)$ followed by ENTER. This will furnish the sum of the next to the last answer on the HOME SCREEN followed by the last answer on the HOME SCREEN. Keep pressing ENTER to generate new terms of this sequence. See **FIGURES 2-7:**

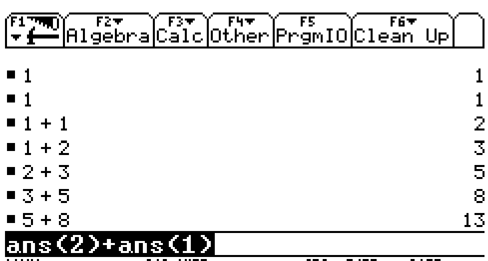

FIGURE 2

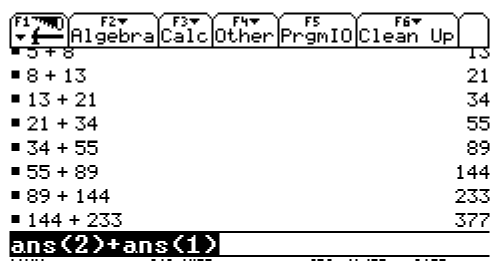

FIGURE 3

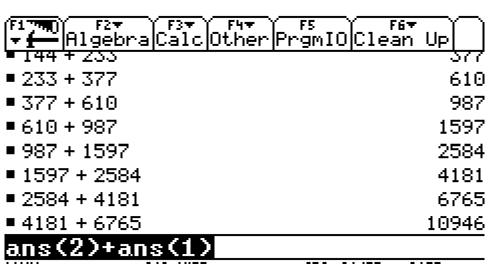

FIGURE 4

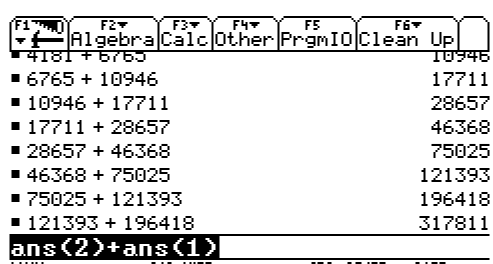

FIGURE 5

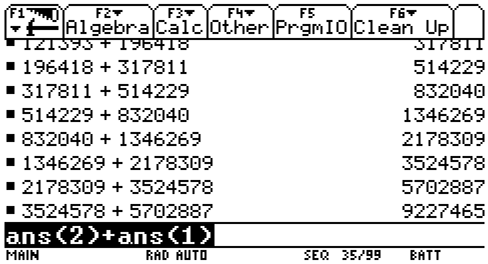

FIGURE 6

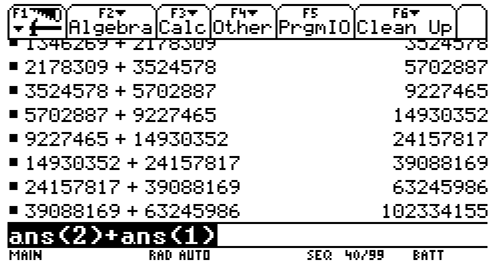

FIGURE 7

If one reads this data, they see two numbers on the bottom right; for example in **FIGURE 5,** one sees 28/99. The 28th term is the last answer in **FIGURE 5** and is 317811. There are 99 possible answers retained by the calculator. One can adjust this last number. From the HOME SCREEN, use the keystrokes F1 9: Format (see **FIGURE 8**) and press ENTER. You will see what is called History Pairs and use the right arrow cursor to see the choices, which indicate the number of answers one can recover from the HOME SCREEN (see **FIGURES 9-10**). The factory setting for the History Pairs is 30.

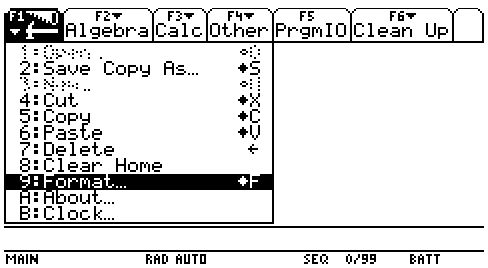

FIGURE 8

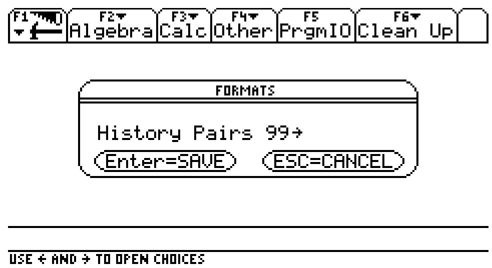

FIGURE 9

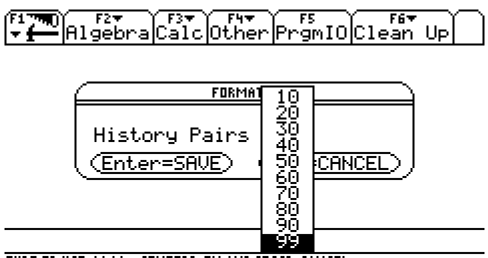

FIGURE 10

Based on the data in **FIGURES 2-7,** we conjecture that every fourth Fibonacci integer is divisible by three.

$$\left\{\begin{array}{l} F_4 = 3, F_8 = 21, F_{12} = 144, F_{16} = 987, F_{20} = 6765 \\ F_5 = 5, F_{10} = 55, F_{15} = 610, F_{20} = 6765, F_{25} = 75025 \\ F_8 = 21, F_{16} = 987, F_{24} = 46368, F_{32} = 2178309, F_{40} = 102334155 \end{array}\right\}.$$

Proceeding to SEQUENCE GRAPHING (use the keystrokes MODE followed by the right arrow cursor to option 4: SEQUENCE followed by ENTER), we see an SEQ at the bottom of the HOME SCREEN. See **FIGURES 11-12:**

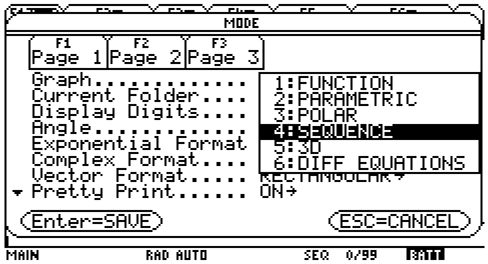

FIGURE 11

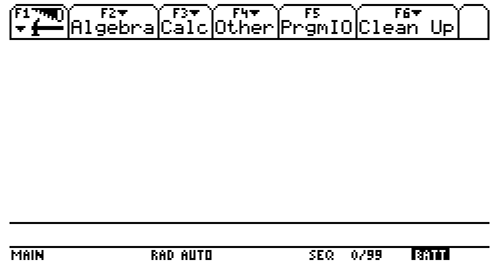

FIGURE 12

Next proceed to the Y= EDITOR and input the following as in **FIGURE 13** with the Standard Viewing Window, Graph, Table Setup, and a portion of the TABLE in **FIGURES 14-20:**

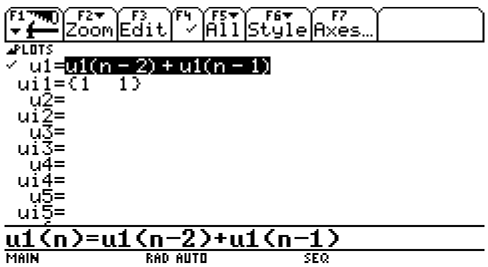

FIGURE 13

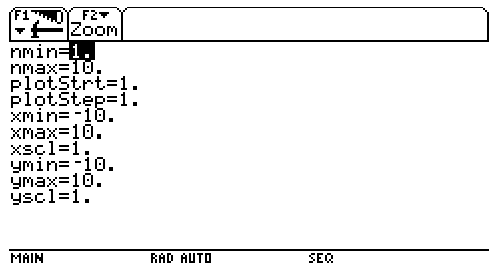

FIGURE 14

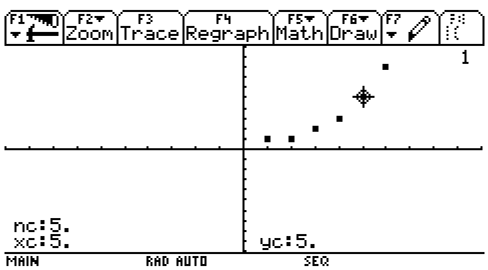

FIGURE 15

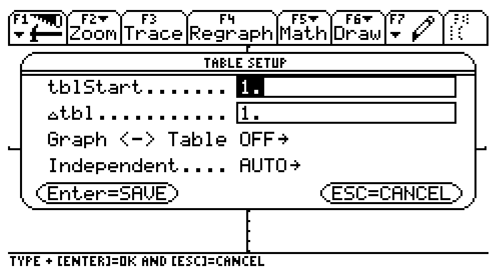

FIGURE 16

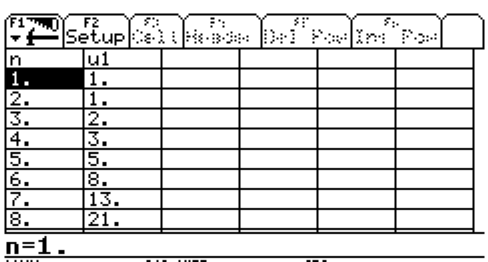

FIGURE 17

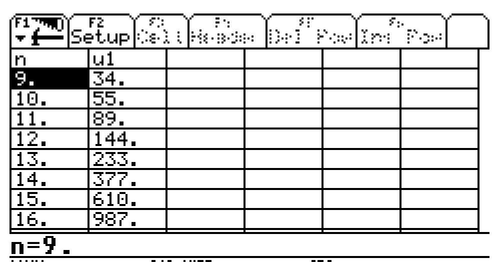

FIGURE 18

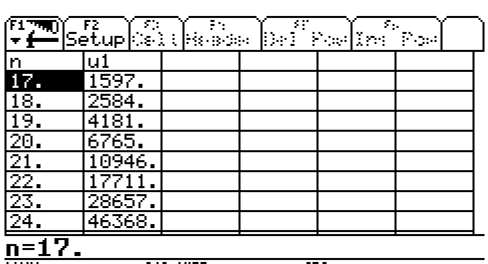

FIGURE 19

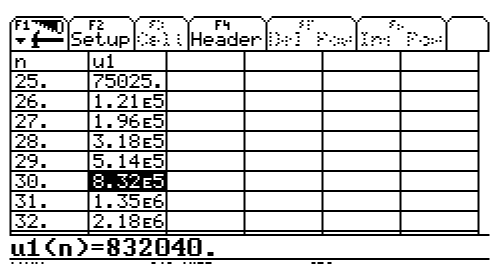

FIGURE 20

Some Comments on the above screen captures:

1. **In FIGURE 13,** note that the recursion rule is provided on the line headed by $u1$ while the line headed by $ui1$ records the initial two terms of the sequence, the second followed by the first. There is no comma between the two 1's in Pretty Print although one separates the two initial 1's with a comma on the entry line. See **FIGURE 21:**

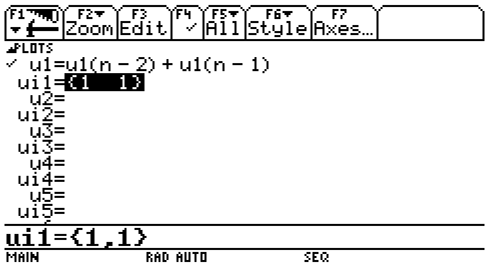

FIGURE 21

2. Note from **FIGURE 15** that 5 is the fifth term of the Fibonacci sequence.
3. Since a sequence is a function whose domain is the set of positive integers, the Tbl Start begins at 1 in **FIGURE 16.**
4. Only five figures are possible in any cell. Thus all terms of the Fibonacci sequence after the twenty-fifth are expressed in scientific notation. If one places their cursor on the output value, however, the exact value is determined as in **FIGURE 20** where the thirtieth term is given exactly as 832040.

Thus if one considers the famous Fibonacci sequence or any Fibonacci-like sequence (that is a sequence whose first two terms can be anything one pleases but each term thereafter follows the recursion rule in the Fibonacci sequence), form the sum of any six consecutive terms and divide this sum by four. We do this for three separate sets and form a conjecture. The results are tabulated in the following TABLE:

SUM OF SIX CONSECUTIVE FIBONACCI NUMBERS:	SUM OF THE MEMBERS OF THE SET:	QUOTIENT WHEN THE SUM IS DIVIDED BY 4:
$\{2,3,5,8,13,21\}$	52	13…FIFTH TERM
$\{1,1,2,3,5,8\}$	20	5…FIFTH TERM
$\{55,89,144,233,377,610\}$	1508	377…FIFTH TERM

<u>CONJECTURE:</u> The sum of any six consecutive Fibonacci numbers is divisible by 4 and the quotient will always be the fifth term in the sequence.

<u>Proof:</u> Consider the initial two terms of the Fibonacci sequence to be $x\ and\ y$. The six consecutive terms of the sequence are as follows: $\{x, y, x+y, x+2\cdot y, 2\cdot x+3\cdot y, 3\cdot x+5\cdot y\}$. We employ the VOYAGE 200 to form the sum and divide the resulting sum by 4. See **FIGURE 21:**

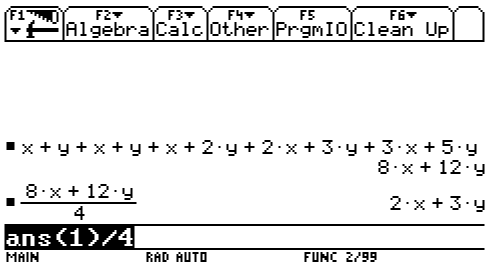

FIGURE 21

Let us next form the sum of any ten consecutive integers and divide this sum by 11. View this for three separate sets and form a conjecture. The results are tabulated below:

View this for three separate sets and form a conjecture. The results are tabulated below: SUM OF TEN CONSECUTIVE FIBONACCI NUMBERS:	SUM OF THE MEMBERS OF THE SET:	QUOTIENT WHEN THE SUM IS DIVIDED BY 11:
$\{2,3,5,8,13,21,34,55,89,144\}$	374	34…SEVENTH TERM
$\{1,1,2,3,5,8,13,21,34,55\}$	143	13…SEVENTH TERM
$\{55,89,144,233,377,610,987,1597,2584,4181\}$	10857	987…SEVENTH TERM

<u>CONJECTURE:</u> The sum of any ten consecutive Fibonacci numbers is divisible by 11 and the quotient will always be the seventh term in the sequence.

<u>Proof:</u> Consider the initial two terms of the Fibonacci sequence to be $x\ and\ y$. The ten consecutive terms of the sequence are as follows:
$\{x, y, x+y, x+2\cdot y, 2\cdot x+3\cdot y, 3\cdot x+5\cdot y, 5\cdot x+8\cdot y, 8\cdot x+13\cdot y, 13\cdot x+21\cdot y, 21\cdot x+34\cdot y\}$ Let us employ the TI-89 to form the sum and divide the resulting sum by 11. See **FIGURES 22-25:**

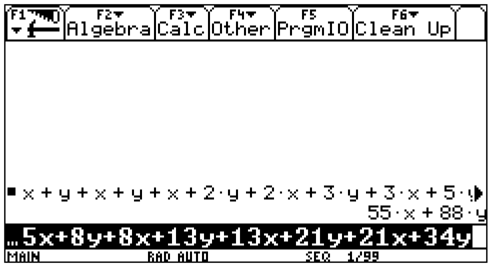

FIGURE 22

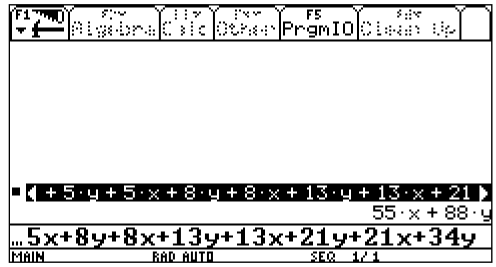

FIGURE 23

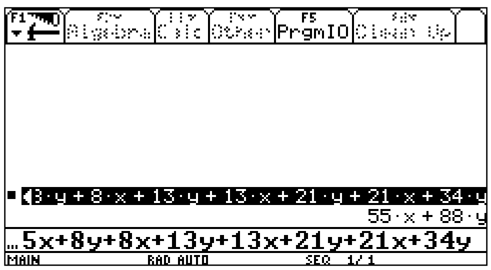

FIGURE 24

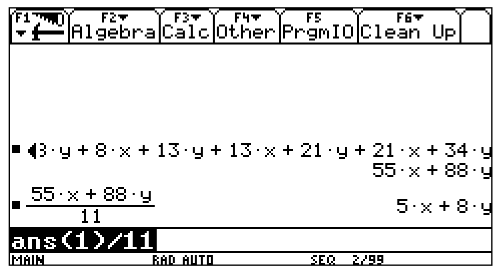

FIGURE 25

Notice $5\cdot x+8\cdot y$ is the seventh term in the sequence which is a neat Fibonacci number trick.

Let us next form the sum of any fourteen consecutive integers and divide this sum by 29 for three separate sets and form a conjecture. The results are tabulated below:

SUM OF FOURTEEN CONSECUTIVE FIBONACCI NUMBERS:	SUM OF THE MEMBERS OF THE SET:	QUOTIENT WHEN THE SUM IS DIVIDED BY 29:
$\{2,3,5,8,13,21,34,55,89,144,\ 233,\ 377,\ 610,\ 987\}$	2581	89…NINTH TERM
$\{1,1,2,3,5,8,13,21,34,55,\ 89,\ 144,\ 233,\ 377\}$	986	34…NINTH TERM
$\left\{\begin{matrix}55,89,144,233,377,610,987,1597,2584,\\ 4181,6765,\ 10946,\ 17711,\ 28657\end{matrix}\right\}$	74936	2584…NINTH TERM

<u>**CONJECTURE:**</u> **The sum of any fourteen consecutive Fibonacci numbers is divisible by 29 and the quotient will always be the ninth term in the sequence.**

<u>**Proof:**</u> Consider the initial two terms of the Fibonacci sequence to be x *and* y. The fourteen consecutive terms of the sequence are as follows:

$$\left\{\begin{matrix}x,\ y,\ x+y,\ x+2\cdot y,\ 2\cdot x+3\cdot y,\ 3\cdot x+5\cdot y,\ 5\cdot x+8\cdot y,\ 8\cdot x+13\cdot y,\ 13\cdot x+21\cdot y,\ 21\cdot x+34\cdot y,\\ 34\cdot x+55\cdot y,\ 55\cdot x+89\cdot y,\ 89\cdot x+144\cdot y,\ 144\cdot x+233\cdot y\end{matrix}\right\}$$

The VOYAGE 200 is used to form the sum and divide the total by 29. See **FIGURES 26-30:**

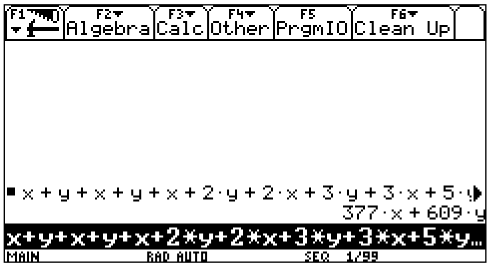

FIGURE 26

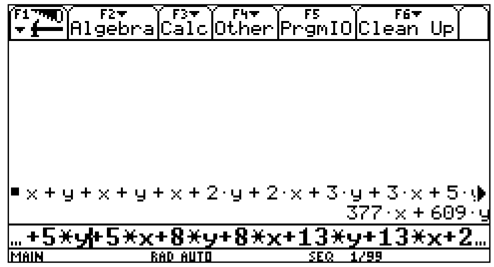

FIGURE 27

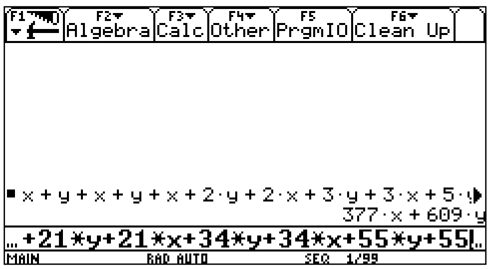

FIGURE 28

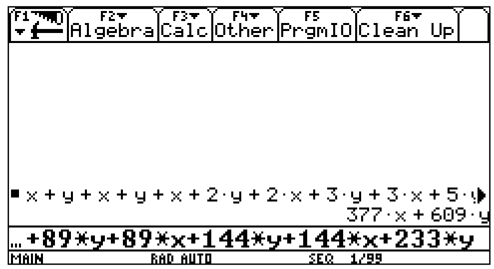

FIGURE 29

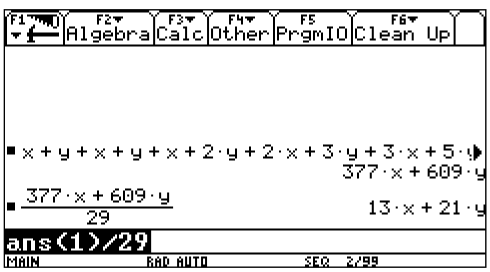

FIGURE 30

II. If E represents the set of even integers and O connotes the set of odd integers, then empirical evidence for the four possible combinations of addition scenarios:
$E+E=E, E+O=O, O+E=O, \textit{and } O+O=E$ and the four possible combinations of multiplication scenarios: $E \cdot E = E, E \cdot O = E, O \cdot E = E, \textit{and } O \cdot O = O.$ See **FIGURES 31-38** respectively below with the aid of The VOYAGE 200 CAS Calculator from Texas Instruments:

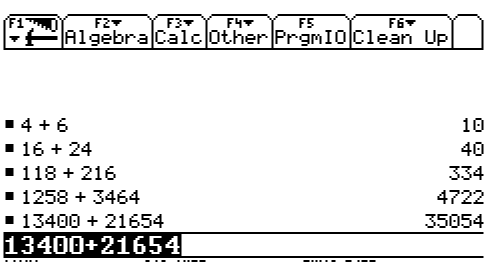

FIGURE 31

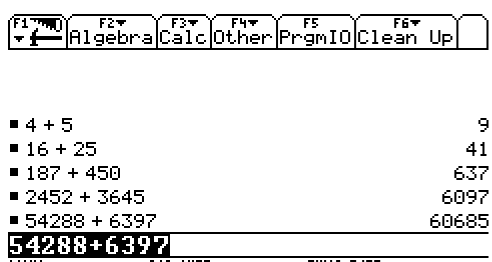

FIGURE 32

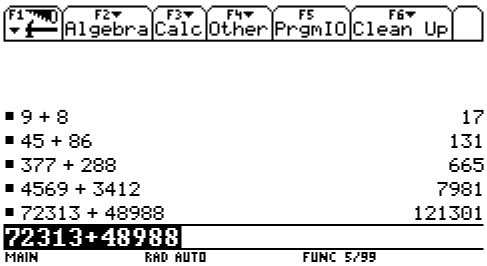

FIGURE 33

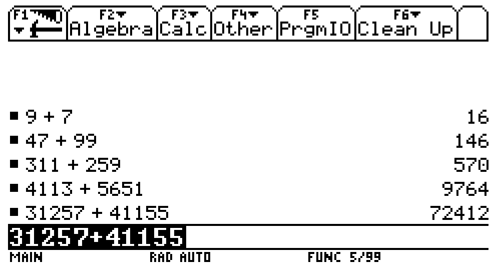

FIGURE 34

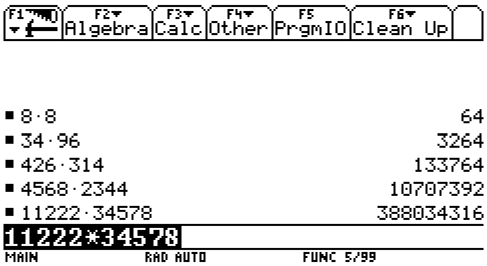

FIGURE 35

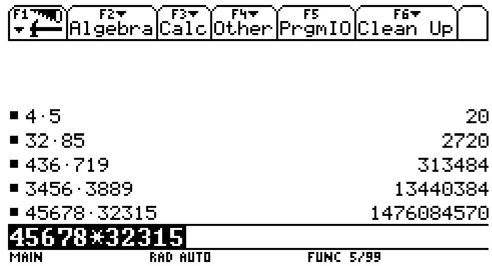

FIGURE 36

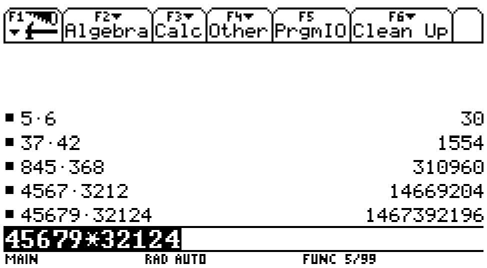

FIGURE 37

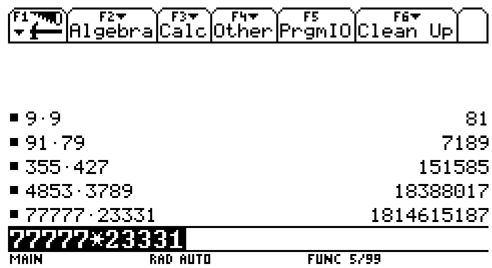

FIGURE 38

Based on reasoning inductively to a general conclusion via the observations of five specific cases, one can conjecture the following displayed in **TABLES 1 and 2:**

+	E	O
E	E	O
O	O	E

TABLE 1

×	E	O
E	E	E
O	E	O

TABLE 2

To furnish a formal proof, we appeal to the definitions of even and odd integers:

We will verify each of the four cases.

(i). To prove that $E+E=E$, let $a,b\in E$. Then $\exists k,l\in Z \ni (s.t.)\, a=2\cdot k(10)\ and\ b=2\cdot l(11)$. Now $a+b=2\cdot k+2\cdot l=2\cdot(k+l)$. $k,l\in\mathbb{Z}\Rightarrow k+l\in\mathbb{Z}$. Call $k+l=s.$ Hence $a+b=2\cdot s\Rightarrow a+b\in E.$ □

(ii). To prove $O+E=O$, let $a\in O\ and\ b\in E$. One can secure $k,l\in\mathbb{Z}\ni a=2\cdot k+1(12)\ and\ b=2\cdot l(13)$.

Next note that $a+b=2\cdot k+2\cdot l+1=(2\cdot k+2\cdot l)+1=2\cdot(k+l)+1.$ Observe that $k,l\in\mathbb{Z}\Rightarrow k+l\in\mathbb{Z}$. Denote $k+l=s$. Hence $a+b=2\cdot s+1\Rightarrow a+b\in O.$ □

(iii). To show that $E\times O=E$, let $a\in E\ and\ b\in O$. One can thus find $k,l\in\mathbb{Z}\ni a=2\cdot k(13)\ and\ b=2\cdot l+1(14)$.

$a\cdot b=(2\cdot k)\cdot(2\cdot l+1)=2\cdot k\cdot 2\cdot l+2\cdot k\cdot 1=4\cdot k\cdot l+2\cdot k=2\cdot(2\cdot k\cdot l+k)$. Appealing to the facts that $k,l\in\mathbb{Z}\Rightarrow k+l\in\mathbb{Z}\ and\ k\cdot l\in\mathbb{Z}$, we obtain $2\cdot k\cdot l+k\in\mathbb{Z}$. (Recall that $2\in\mathbb{Z}$.) Call $2\cdot k\cdot l+k=u$. Hence $a\cdot b=2\cdot u\Rightarrow a\cdot b\in E.$ □

(iv). To prove our final case and demonstrate that $O\times O=O$, let $a,b\in O$. Then by definition, $\exists k,l\in\mathbb{Z}\ni a=2\cdot k+1(15)\ and\ b=2\cdot l+1(16)$. Now

$a\cdot b=(2\cdot k+1)\cdot(2\cdot l+1)=4\cdot k\cdot l+2\cdot k+2\cdot l+1=(4\cdot k\cdot l+2\cdot k+2\cdot l)+1=2\cdot(2\cdot k\cdot l+k+l)+1.$ Employing the closure properties of $+\ and\ \times\ in\ \mathbb{Z}$, one is assured that $2\cdot k\cdot l+k+l\in\mathbb{Z}$. Call $2\cdot k\cdot l+k+l=v$. Thus $a\cdot b=2\cdot v+1\Rightarrow a\cdot b\in O.$ □

III. We know that the areas and perimeters of rectangles are given by the respective formulas $A=l\cdot w\ and\ P=2\cdot l+2\cdot w=2\cdot(l+w)$; $l=length\ and\ w=width.$ A student in the earlier grades might form tables such as the following for the respective perimeters of 20 and 100:

l:	w:	P:	A:
10	0	20	0
9	1	20	9
8	2	20	16
7	3	20	21
6	4	20	24
5	5	20	25
4	6	20	24

3	7	20	21
2	8	20	16
1	9	20	9
0	10	20	0

l:	w:	P:	A:
16	0	32	0
15	1	32	15
14	2	32	28
13	3	32	39
12	4	32	48
11	5	32	55
10	6	32	60
9	7	32	63
8	8	32	64
7	9	32	63
6	10	32	60
5	11	32	55
4	12	32	48
3	13	32	39
2	14	32	28
1	15	32	15
0	16	32	0

Based upon the outcomes in the table, it appears that the area of the largest rectangles having respective perimeters of 20 and 32 are the respective 5×5 *and* 8×8 squares. Is this always the case; namely that the area of largest rectangle having a given perimeter is necessarily a square whose length is one-quarter of the perimeter? Stay tuned. Progressing along, the student of algebra can graph the equations for the area utilizing algebra. For example, if $P=20,\ \textit{then}\ 2\cdot(l+w)=20\Rightarrow l+w=10\Rightarrow w=10-l\Rightarrow A=l\cdot w\Rightarrow A=l\cdot(10-l)\Rightarrow A=10\cdot l-l^2$. Using the TI-84 or the TI-89, one can graph this equation and secure the maximum area (see **FIGURES 39-48):**

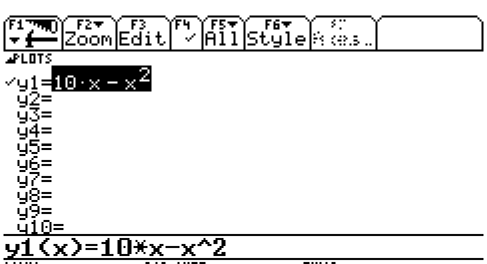

FIGURE 39

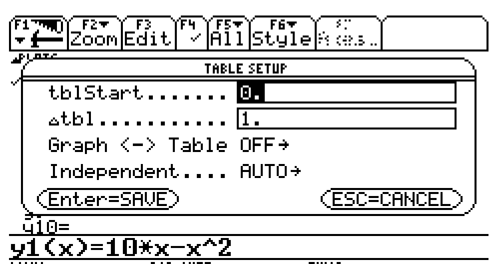

FIGURE 40

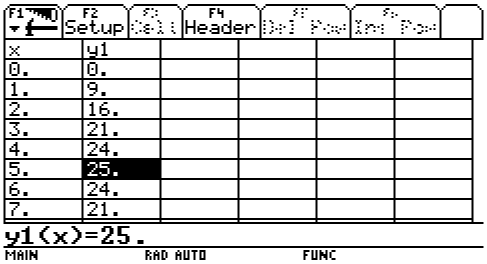

FIGURE 41

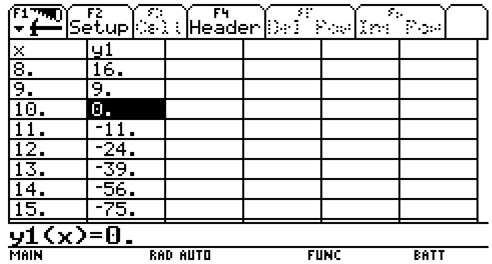

FIGURE 42

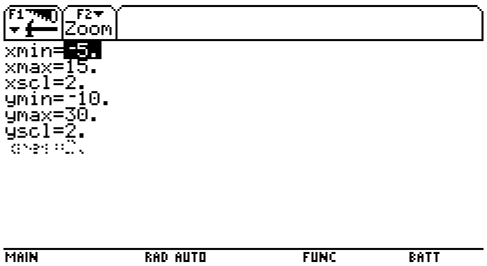

FIGURE 43

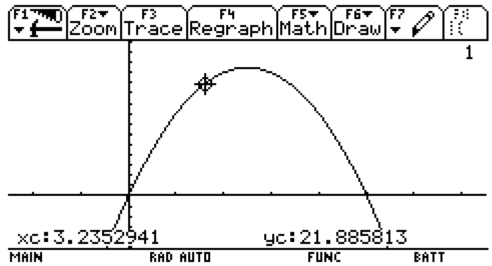

FIGURE 44

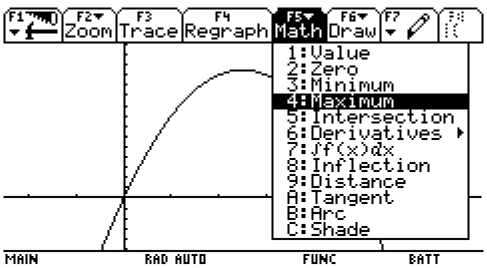

FIGURE 45

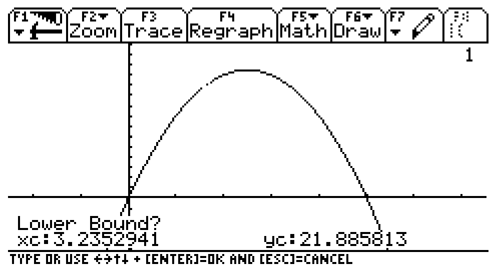

FIGURE 46

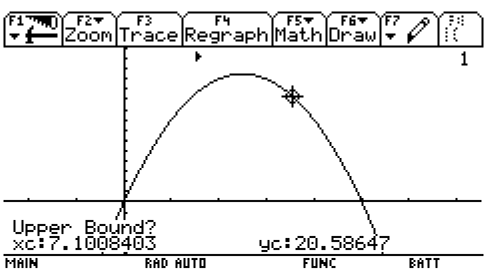

FIGURE 47

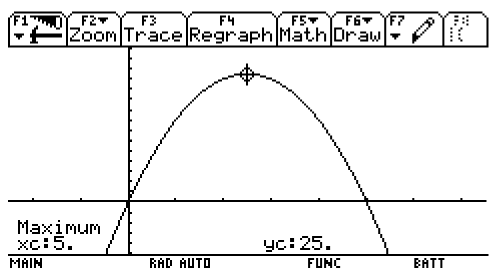

FIGURE 48

On the other hand, if

$P=32,\ \textit{then}\ 2\cdot(l+w)=32 \Rightarrow l+w=16 \Rightarrow w=16-l \Rightarrow A=l\cdot w \Rightarrow A=l\cdot(16-l) \Rightarrow A=16\cdot l-l^2.$

Using the TI-84 or the TI-89, one can graph this equation and secure the maximum area (see **FIGURES 49-59):**

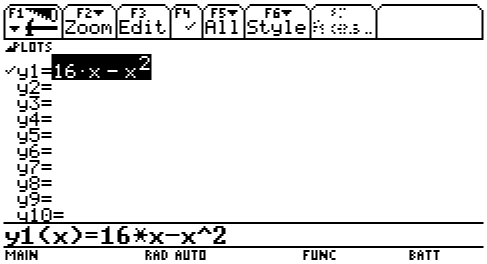

FIGURE 49

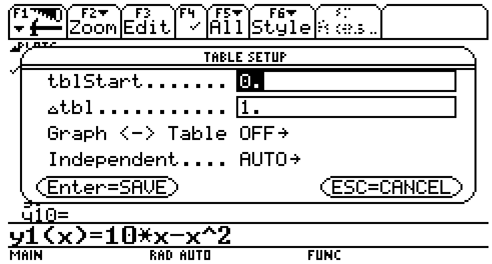

FIGURE 50

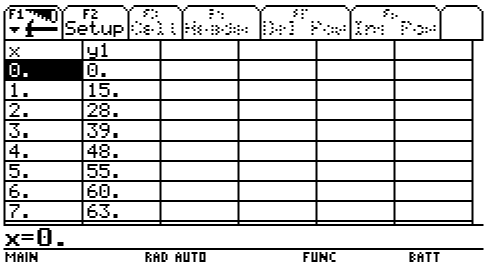

FIGURE 51

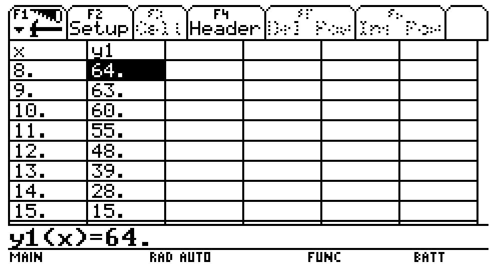

FIGURE 52

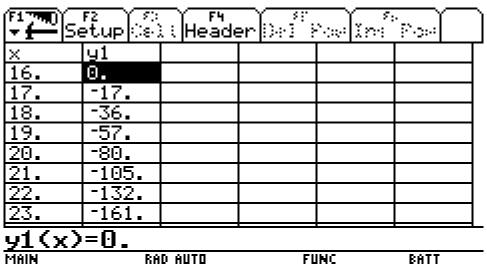

FIGURE 53

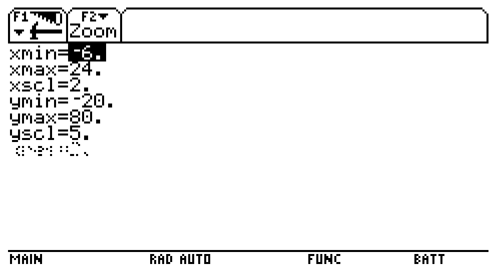

FIGURE 54

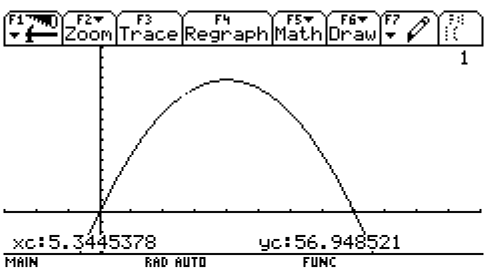

FIGURE 55

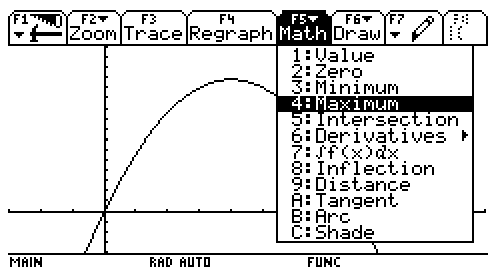

FIGURE 56

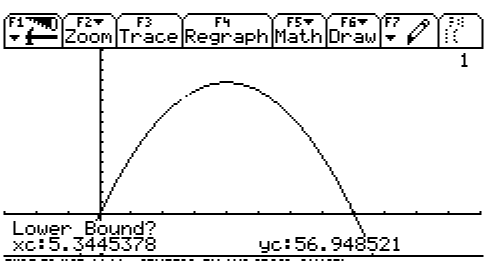

FIGURE 57

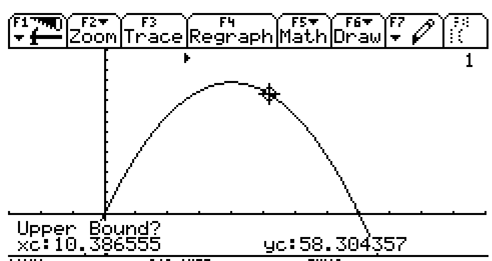

FIGURE 58

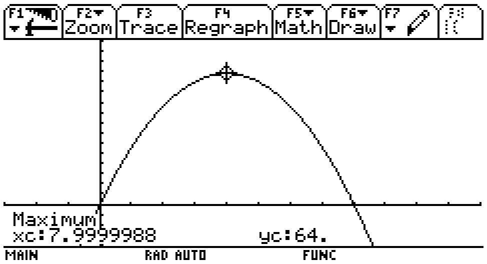

FIGURE 59

Observe that there are constraints on l *and* w; *namely that* $0 \le l \le \frac{P}{2}$ *and* $0 \le w \le \frac{P}{2}$; for otherwise the other dimension would be negative. Hence for rectangles having perimeters 20 and 32 respectively, the lengths and widths could not exceed 10 and 16 respectively. We still have not resolved the general question; namely what is the area of the largest rectangle having a given perimeter P? We prove using calculus that one always obtains a square each of whose sides has length $l = \frac{P}{4}$.

$A = l \cdot w \; (1)$ *and* $P = 2 \cdot l + 2 \cdot w. \; (2)$ The area formula represents the primary equation; for we are seeking to maximize this quantity and the perimeter formula which serves to aid us in our task is the secondary equation. Solving (2) for w and substituting into (1), we obtain

$$P = 2 \cdot l + 2 \cdot w \Rightarrow P - 2 \cdot l = 2 \cdot w \Rightarrow \frac{P - 2 \cdot l}{2} = \frac{P}{2} - l = w \Rightarrow A(l) = l \cdot \left(\frac{P}{2} - l\right) = \frac{P \cdot l}{2} - l^2. \; (3)$$

Differentiating (3) with respect to l and setting $\frac{dA}{dl} = 0,$ we obtain

$\frac{dA}{dl} = \frac{P}{2} - 2 \cdot l$ *and* $\frac{dA}{dl} = 0 \Leftrightarrow \frac{P}{2} - 2 \cdot l = 0 \Leftrightarrow \frac{P}{2} = 2 \cdot l \Leftrightarrow \frac{P}{4} = l.$ $l = \frac{P}{4}$ represents the critical number of the area function $A(l)$. Using the second derivative test for relative extrema, we note that $\frac{d^2A}{dl^2} = -2 < 0 \Rightarrow l = \frac{P}{4}$ leads to a relative maximum of the area function $A(l)$. Substituting this value into (2), one obtains $A\left(\frac{P}{4}\right) = \left(\frac{P}{4}\right) \cdot \left(\frac{P}{2} - \frac{P}{4}\right) = \left(\frac{P}{4}\right) \cdot \left(\frac{P}{4}\right) = \frac{P^2}{16}$. To see that this is the largest possible area, note that since $A(l)$ is a polynomial function and hence continuous over $\mathbb{R}$ and hence over $\left[0, \frac{P}{2}\right] \subseteq \mathbb{R}$, the Extreme Value Theorem guarantees the existence of both an absolute maximum M and an absolute minimum m somewhere over $\left[0, \frac{P}{2}\right]$. We use the Tabular Method below to locate the absolute extrema where the largest of the tabulated values represents the absolute maximum of the area function over the interval and the smallest of the tabulated values represents the absolute minimum of the area function over the interval:

l :	$A(l)=\dfrac{P\cdot l}{2}-l^2$:
0	0
$\dfrac{P}{4}$	$\dfrac{P^2}{16}$
$\dfrac{P}{2}$	0

Note that $A\left(\dfrac{P}{2}\right)=\dfrac{P\cdot\left(\dfrac{P}{2}\right)}{2}-\left(\dfrac{P}{2}\right)^2=\dfrac{\dfrac{P^2}{2}}{2}-\dfrac{P^2}{4}=\dfrac{P^2}{4}-\dfrac{P^2}{4}=0.$ Hence the maximum area is obtained when one has a $\dfrac{P}{4}\times\dfrac{P}{4}$ square.

IV. If we add successive unit fractions with even denominators, starting with $\dfrac{1}{2}$, we obtain the following for fifteen iterations in **FIGURES 60-62:**

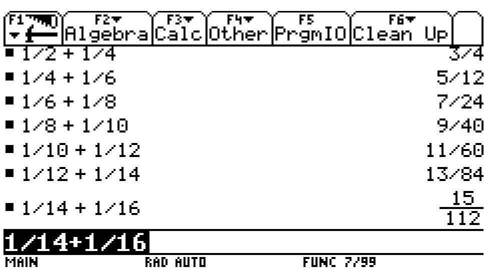

FIGURE 60

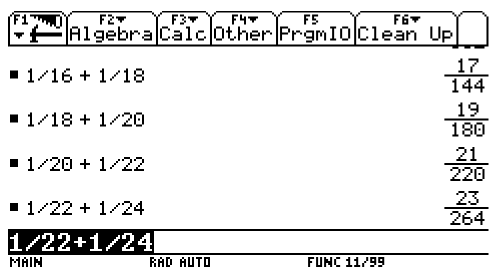

FIGURE 61

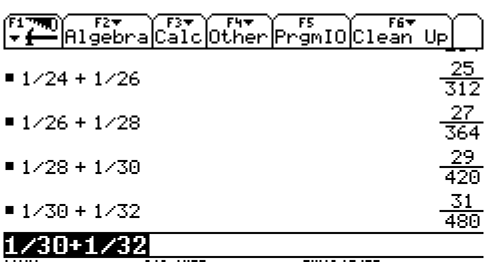

FIGURE 62

Similarly if we add successive unit fractions with odd denominators starting with 1/3, we obtain the following for fifteen iterations in **FIGURES 63-65:**

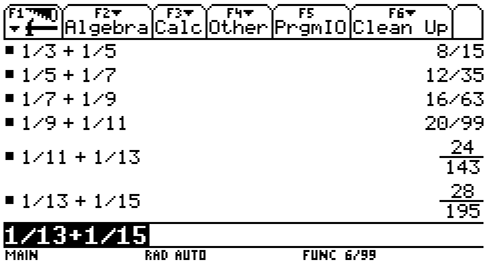

FIGURE 63

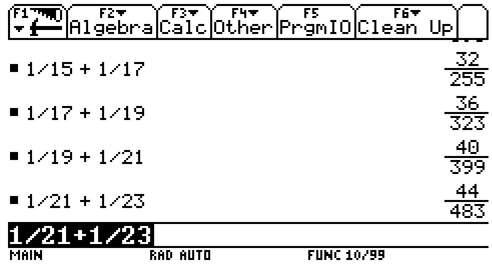

FIGURE 64

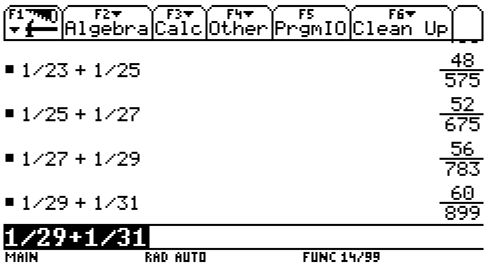

FIGURE 65

Consider the numerators and denominators of the sums obtained in **FIGURES 60-65.** We find that all form the legs of Primitive Pythagorean Triangles. In **FIGURES 66-75,** we show that the sum of the squares of the numerators and denominators of each of these fractions is a perfect square and hence a Pythagorean Triple is formed. Moreover, since $(a,b,c)=1$ in the sense that there are no common integer factors other than 1 among the components, the triples are classified as Primitive Pythagorean triples.

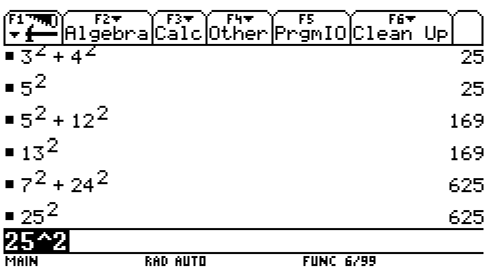

FIGURE 66

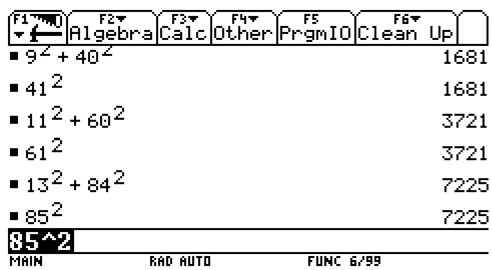

FIGURE 67

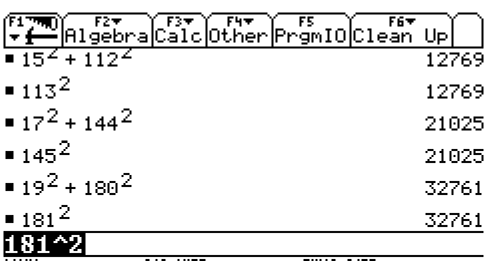

FIGURE 68

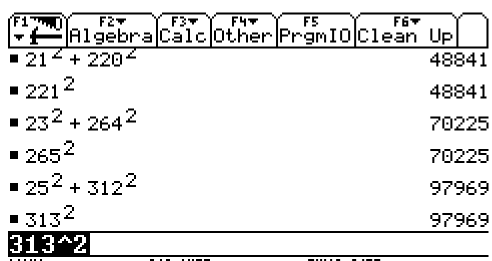

FIGURE 69

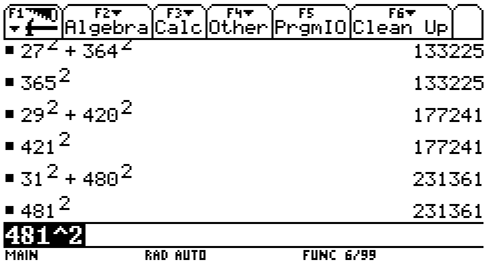

FIGURE 70

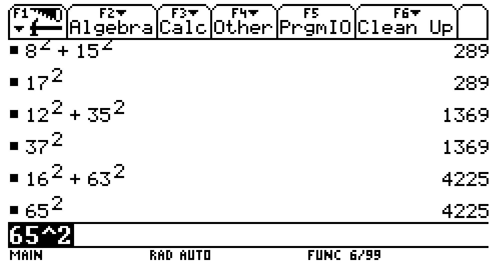

FIGURE 71

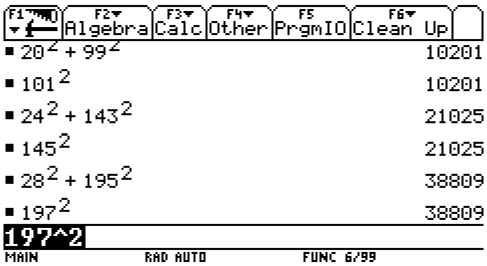

FIGURE 72

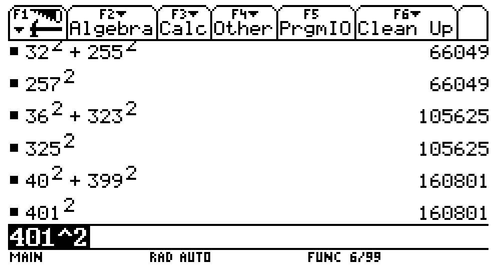

FIGURE 73

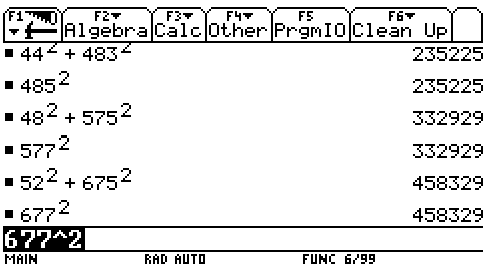

FIGURE 74

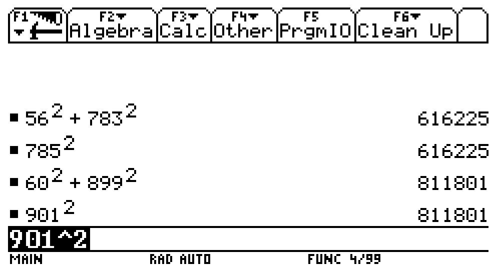

FIGURE 75

Are these results always true and are all Primitive Pythagorean Triples obtained in this manner? The answers are YES and NO respectively. The latter question can be resolved by noting that the PPT (77, 36, 85) is not generated by this process. See **FIGURE 76** below. See **FIGURES 78-81** for a general proof considering the cases of the sum of consecutive even unit fractions and the sum of consecutive odd unit fractions separately with **FIGURE 77** displaying the relevant Algebra Menu F2 on the TI-89/VOYAGE 200.

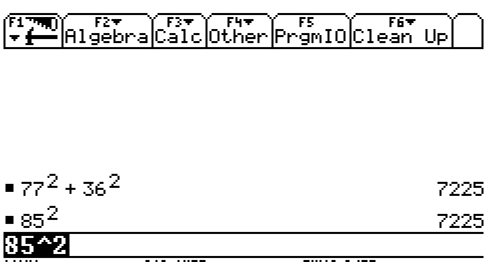

FIGURE 76

One can show that the following is true in general. If we consider that any even integer is of the form $2 \cdot m$ for some integer m and any odd integer is of the form $2 \cdot n + 1$ for some integer n, then we have the following for the sum of two unit fractions with consecutive even denominators:

$\dfrac{1}{2\cdot m}+\dfrac{1}{2\cdot m+2}=\dfrac{2\cdot m+1}{2\cdot m^2+2\cdot m}$. Taking the sum of the squares of the numerator and denominator of this fraction, we note

$(2\cdot m+1)^2+\left(2\cdot m^2+2\cdot m\right)^2=4\cdot m^4+8\cdot m^3+8\cdot m^2+4\cdot m+1=\left(2\cdot m^2+2\cdot m+1\right)^2$. Hence one obtains the Primitive Pythagorean triple $\left(2\cdot m+1,\ 2\cdot m^2+2\cdot m,\ 2\cdot m^2+2\cdot m+1\right)$.

For the sum of two unit fractions with consecutive odd denominators, we observe the following:

$\dfrac{1}{2\cdot n+1}+\dfrac{1}{2\cdot n+3}=\dfrac{4\cdot n+4}{4\cdot n^2+8\cdot n+3}$. Taking the sum of the squares of the numerator and denominator of this fraction, we note that

$(4\cdot n+4)^2+\left(4\cdot n^2+8\cdot n+3\right)^2=16\cdot n^4+64\cdot n^3+104\cdot n^2+80\cdot n+25=\left(4\cdot n^2+8\cdot n+5\right)^2$. Thus one obtains the Primitive Pythagorean Triple $\left(4\cdot n+4,\ 4\cdot n^2+8\cdot n+3,\ 4\cdot n^2+8\cdot n+5\right)$.

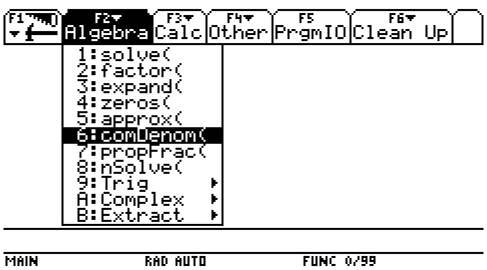

FIGURE 77

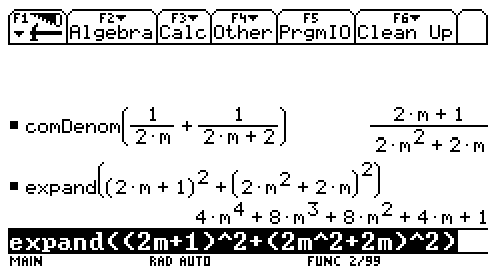

FIGURE 78

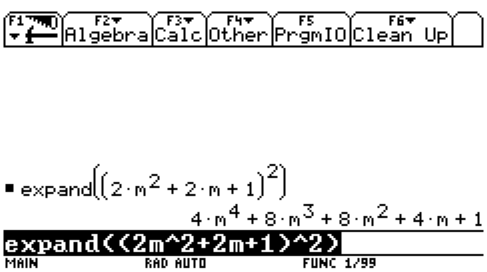

FIGURE 79

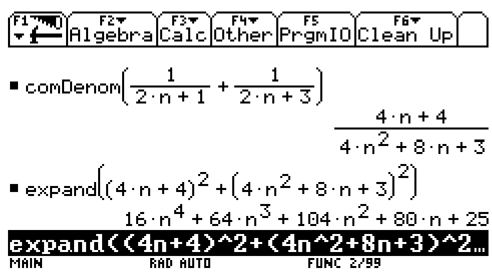

FIGURE 80

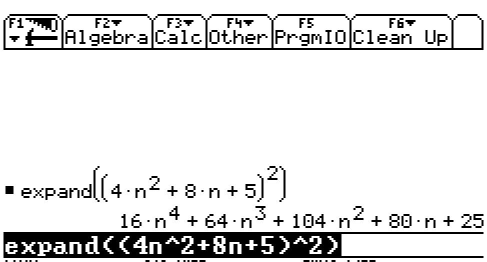

FIGURE 81

V. For your convenience, here are the first five rows of Pascal's Triangle:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Based on the initial five rows of Pascal's Triangle, the next five rows are displayed. Let us write Rows 0-10 as follows:

1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	**5**	10	10	5	1					
1	**6**	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	120	252	210	120	45	10	1

In the following table, we determine the parity of the entries in the initial twenty-five rows of Pascal's Triangle where O and E denote the respective odd and even numbered entries.

O																								
O	O																							

O	E	O																						
O	O	O	O																					
O	E	E	E	O																				
O	O	E	E	O	O																			
O	E	O	E	O	E	O																		
O	O	O	O	O	O	O	O																	
O	E	E	E	E	E	E	E	O																
O	O	E	E	E	E	E	E	O	O															
O	E	O	E	E	E	E	E	O	E	O														
O	O	O	O	E	E	E	E	O	O	O	O													
O	E	E	E	O	E	E	E	O	E	E	E	O												
O	O	E	E	O	O	E	E	O	O	E	E	O	O											
O	E	O	E	O	E	O	E	O	E	O	E	O	E	O										
O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O									
O	E	E	E	E	E	E	E	E	E	E	E	E	E	E	E	O								
O	O	E	E	E	E	E	E	E	E	E	E	E	E	E	E	O	O							
O	E	O	E	E	E	E	E	E	E	E	E	E	E	E	E	O	E	O						
O	O	O	O	E	E	E	E	E	E	E	E	E	E	E	E	O	O	O	O					
O	E	E	E	O	E	E	E	E	E	E	E	E	E	E	E	O	E	E	E	O				
O	O	E	E	O	O	E	E	E	E	E	E	E	E	E	E	O	O	E	E	O	O			
O	E	O	E	O	E	O	E	E	E	E	E	E	E	E	E	O	E	O	E	O	E	O		
O	O	O	O	O	O	O	O	E	E	E	E	E	E	E	E	O	O	O	O	O	O	O	O	
O	E	E	E	E	E	E	E	O	E	E	E	E	E	E	E	O	E	E	E	E	E	E	E	O

Our next goal is to tabulate in each row of the triangle constructed above the numbers of odd entries, even entries, total entries, the cumulative number of odd entries, even entries and total entries and the ratio of the cumulative number of even and odd entries to the cumulative to the cumulative total. Recall that the initial row is considered row 0. We extend the triangle to the thirty-second row and observe some neat patterns. The analysis of the patterns is then employed to form some nice conjectures.

Row	# of Odds	# of Evens	Total	Cum # of Odds	Cum # of Evens	Cum #	Ratio of Cum Odds to Cum #	Ratio of Cum Evens to Cum #
0	1	0	1	1	0	1	1/1	0/1
1	2	0	2	3	0	3	3/3	0/3
2	2	1	3	5	1	6	5/6	1/6
3	4	0	4	9	1	10	9/10	1/10
4	2	3	5	11	4	15	11/15	4/15
5	4	2	6	15	6	21	15/21	6/21
6	4	3	7	19	9	28	19/28	9/28
7	8	0	8	27	9	36	27/36	9/36
8	2	7	9	29	16	45	29/45	16/45
9	4	6	10	33	22	55	33/55	22/55
10	4	7	11	37	29	66	37/66	29/66
11	8	4	12	45	33	78	45/78	33/78
12	4	9	13	49	42	91	49/91	42/91
13	8	6	14	57	48	105	57/105	48/105
14	8	7	15	65	55	120	65/120	55/120
15	16	0	16	81	55	136	81/136	55/136
16	2	15	17	83	70	153	83/153	70/153
17	4	14	18	87	84	171	87/171	84/171

18	4	15	19	91	99	190	91/190	99/190
19	8	12	20	99	111	210	99/210	111/210
20	4	17	21	103	128	231	103/231	128/231
21	8	14	22	111	142	253	111/253	142/253
22	8	15	23	119	157	276	119/276	157/276
23	16	8	24	135	165	300	135/300	165/300
24	4	21	25	139	186	325	139/325	186/325
25	8	18	26	147	204	351	147/351	204/351
26	8	19	27	155	223	378	155/378	223/378
27	16	12	28	171	235	406	171/406	235/406
28	8	21	29	179	256	435	179/435	256/435
29	16	14	30	195	270	465	195/465	270/465
30	16	15	31	211	285	496	211/496	285/496
31	32	0	32	243	285	528	243/528	285/528
32	2	31	33	245	316	561	245/561	316/561

A number of interesting conjectures can be formed via the analysis of the above table:

1. The entries in rows 1, 3, 7, 15, and 31 are all odd. One can correctly conjecture that the only rows where all the entries are odd are of the form $2^n - 1; n \in \mathbb{N}$.
2. The entries in rows 2, 4, 6, 8, 16, and 32 with the exception of the first and last entries which are both 1 are even. This in general occurs in rows of the form $2^n; n \in \mathbb{N}$ and is an immediate consequence of the first conjecture and the binomial identity $C(n,r) = C(n-1, r-1) + C(n-1, r); r < n.$
3. With the exception of Row 0, the number of odd entries in any row of Pascal's Triangle is always even. This is a consequence of the symmetry of the binomial coefficients as manifested by the identity $C(n,r) = C(n, n-r)$.
4. The number of odd entries in any row of Pascal's Triangle is a power of two.

5. The ratio of the cumulative total of even entries to the total number of entries approaches 1 as the number of rows in the triangle gets large. To cite some examples, the cumulative total of even entries to total entries is 1351/2080 by the time we reach row sixty-four and 6069/8028 when we reach row one hundred twenty-eight.

Based on the entries in Pascal's triangle written in rectangular fashion, the initial five diagonals (left and right) contain the respective entries 1, 1, 1, 1, 1,…; 1, 2, 3, 4, 5,…; 1, 3, 6, 10, 15, 21, 28,…; 1, 4, 10, 20, 35, 56, 84, 120; and 1, 5, 15, 35, 70, 126, 210,… One can find a polynomial model that fits the data with perfect correlation every time! The respective polynomials are $1, n, \frac{n\cdot(n+1)}{2}, \frac{n\cdot(n+1)\cdot(n+2)}{6}$ *and* $\frac{n\cdot(n+1)\cdot(n+2)\cdot(n+3)}{24}$ respectively. In other words, the entries along the first, second, third, fourth and fifth diagonals (left and right) model constant, linear, quadratic, cubic, and quartic polynomials respectively. One can utilize the method of successive (finite) differences to observe that the set of zero, first, second, third and fourth differences are constant respectively. For example, let us consider the entries along the fifth diagonal; namely 1, 5, 15, 35, 70, 126, 210, …. To see that this set of data models a quartic (fourth degree) polynomial, we consider the ordered pairs where the first, second, third, fourth, fifth terms respectively are 1, 5, 15, 35, 70 and 126. This leads to the ordered pairs $(1,1), (2,5), (3,15), (4,35)$ *and* $(5,70)$. One needs five distinct data points to secure a fourth degree polynomial since the form for quartics is $y = f(x) = A\cdot x^4 + B\cdot x^3 + C\cdot x^2 + D\cdot x + E.$ Observe the five unknown constants A, B, C, D *and* E need to be determined.

Sequence:	1		5		15		35		70		126		210
Set of First Differences:		4		10		20		35		56		84	
Set of Second Differences:			6		10		15		21		28		
Set of Third Differences:				4		5		6		7			
Set of Fourth Differences:					1		1		1				

If we choose the first five ordered pairs, this leads to the 5×5 linear system:

$$\begin{aligned}
A + B + C + D + E &= 1\\
16A + 8B + 4C + 2D + E &= 5\\
81A + 27B + 9C + 3D + E &= 15\\
256A + 64B + 16C + 4D + E &= 35\\
625A + 125B + 25C + 5D + E &= 70
\end{aligned}$$

Using the VOYAGE 200 (one could have also used the TI-84), we solve our linear system. See **FIGURES 82-88:**

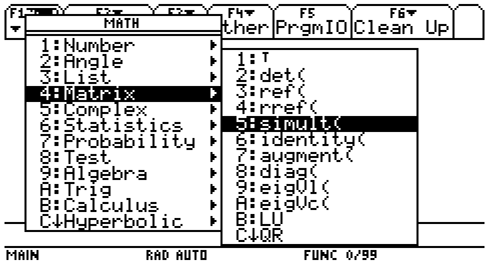

FIGURE 82

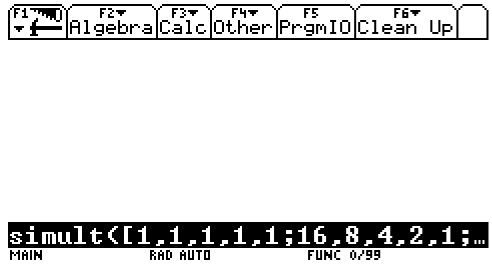

FIGURE 83

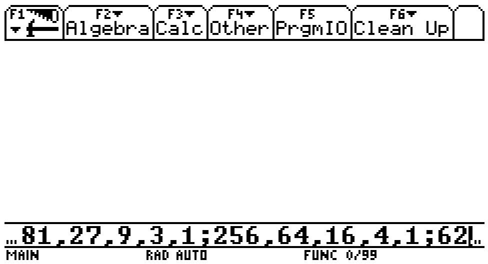

FIGURE 84

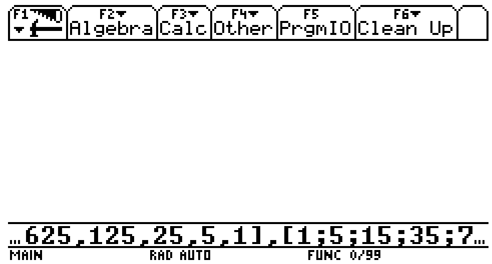

FIGURE 85

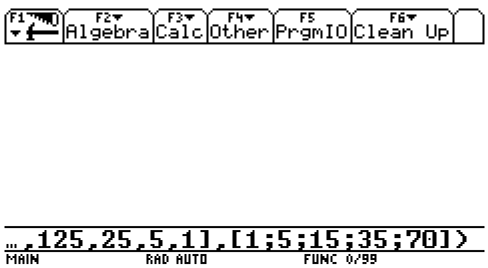

FIGURE 86

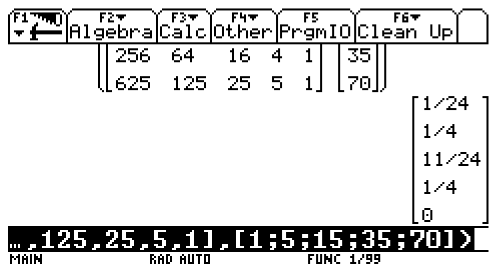

FIGURE 87

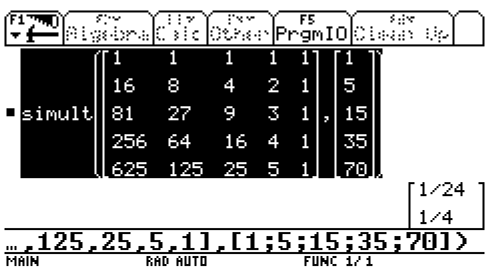

FIGURE 88

Observe that $y = f\left(x\right) = \frac{1}{24}\cdot x^4 + \frac{1}{4}\cdot x^3 + \frac{11}{24}\cdot x^2 + \frac{1}{4}\cdot x$ which fits these data points. If we factor the expression via the VOYAGE 200, we obtain the factored form in **FIGURES 89-90:**

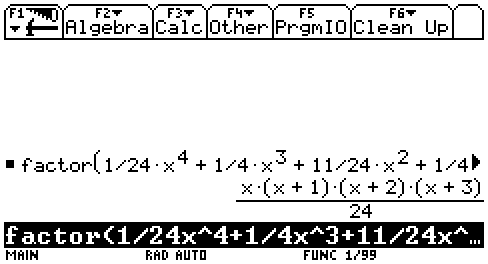

FIGURE 89

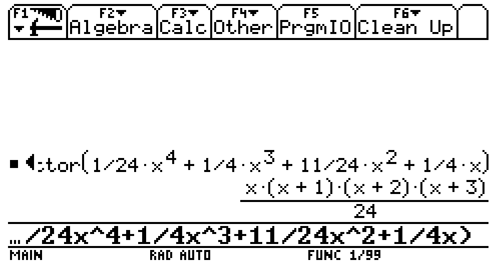

FIGURE 90

In general, one can show by mathematical induction that the entries on the k-th diagonals of Pascal's triangle fit the polynomial model $y = f(x) = \dfrac{x\cdot(x+1)\cdot(x+2)\cdot(x+3)\cdot\ldots(x+k-1)}{k!}$.

We determine the entries in the first five rows of the triangle and both the sum and cumulative sum of the entries in the initial five rows. See **FIGURES 91-96:**

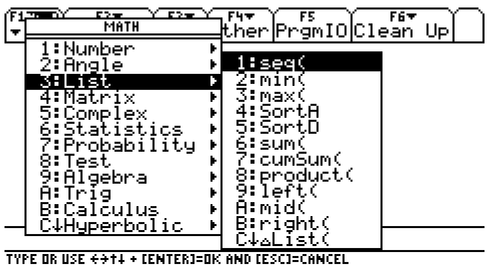

FIGURE 91

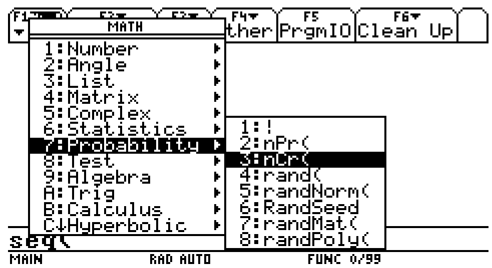

FIGURE 92

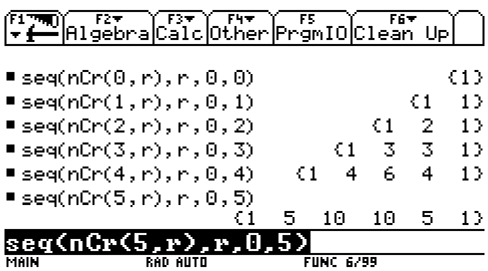

FIGURE 93

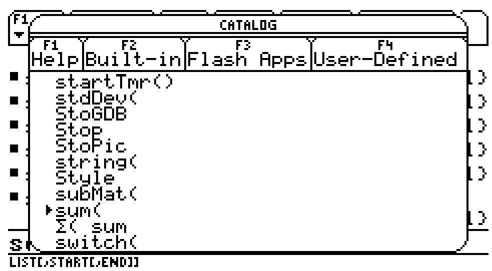

FIGURE 94

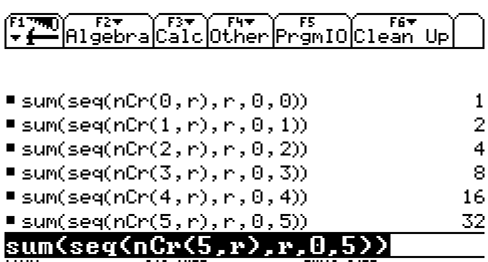

FIGURE 95

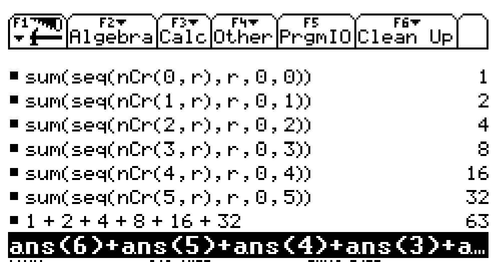

FIGURE 96

VI. Consider the calendar for the month of **MARCH 2015** below:

MARCH 2015

S	M	T	W	R	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28

a. We select several 3×3 groups of numbers and find the sum of these numbers and determine how the obtained sums are related to the middle number.

We first consider the group highlighted in blue above. The integers are 8, 9, 10, 15, 16, 17, 22, 23 and 24 with the middle number being 16. Adding these nine numbers, we obtain $8+9+10+15+16+17+22+23+24=144=9\cdot 16.$

Let us next select a second 3×3 group of numbers highlighted in green below. These numbers are 11, 12, 13, 18, 19, 20, 25, 26 and 27 with 19 serving as the middle number. We note that $11+12+13+18+19+20+25+26+27=171=9\cdot 19.$

MARCH 2015

S	M	T	W	R	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28

We finally select a third 3×3 group of numbers highlighted in red below. These numbers are 12, 13, 14, 19, 20, 21, 26, 27 and 28 where 20 is the middle number. Note that $12+13+14+19+20+21+26+27+28=180=9\cdot 20.$

FEBRUARY 2015

S	M	T	W	R	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28

b. We next prove that the sum of any 9 digits in any 3×3 set of numbers selected from a monthly calendar will always be equal to 9 times the middle number using both algebra and technology to furnish a convincing argument.

Based on our analysis of three specific cases, it seems plausible to conjecture that the sum of any nine elements in the 3×3 group is always nine times the middle number. To show that this is always true, one can employ algebraic reasoning. We let $x=$ the first number in the array. The next numbers are thus $x+1,\ x+2,\ x+7,\ x+8,\ x+9,\ x+14,\ x+15$ *and* $x+16.$ Next note that $x+x+1+x+2+x+7+x+8+x+9+x+14+x+15+x+16=9x+72=9\cdot(x+8)$. Observe that $x+8$ is the median (middle number) in the array completing our proof.

Technology can play a role as well. Let us use a graphing calculator (TI-89) to furnish the specific cases as well as a formal proof. See **FIGURES 97-100:**

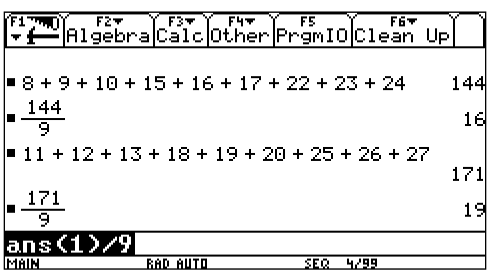

FIGURE 97

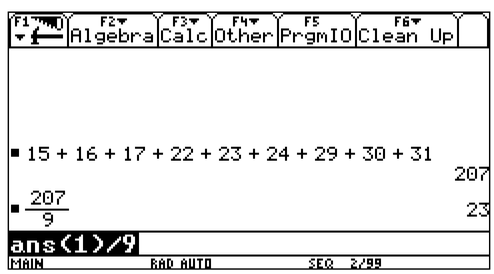

FIGURE 98

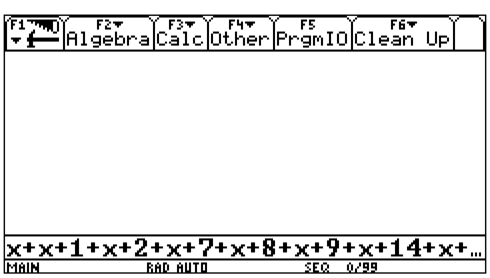

FIGURE 99

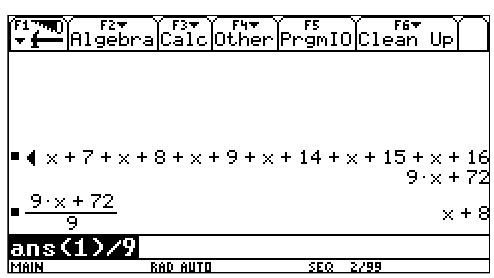

FIGURE 100

VII. We determine if each of the following statements is true or false. If it is true, construct a formal proof. For all false statements, determine an appropriate counterexample and explain what needs to be altered to make the false statements true.

(a). The product of any three integers is divisible by 2 is false. Consider the odd integers 3, 5 and 7. Then $3\cdot 5\cdot 7=105$ *and* $2\nmid 105.$ On the other hand, the product of three consecutive integers is divisible by 2. In addition, the product of three integers, two of which are of even parity is even and hence is divisible by 2. Likewise, the product of three integers such that two are of odd parity is even and hence is divisible by 2.

(b). The sum of four consecutive integers is divisible by 4 is false. Consider the consecutive integers 5, 6, 7 and 8. Then $5+6+7+8=26$ *and* $4\nmid 26.$ Moreover, the sum of four consecutive integers is NEVER divisible by 4. To see this, we appeal to the algebra of remainders. In any string of four consecutive integers, if the first is evenly divisible by four, the second will have a remainder of one upon division by four, the third will have a remainder of two upon division by four and the fourth will have a remainder of three upon division by four. A similar proof can be constructed in the respective cases where the first integer in the sequence has a remainder of one, two and three upon division by 4. Just shift accordingly. Hence the sum of the remainders will be six which when reduced modulo four is two. It is true that the sum of any four consecutive integers is always even and thus divisible by two.

(c). The product of four consecutive integers plus one is always a perfect square is indeed true. Let the consecutive integers be denoted respectively by $n,\ n+1,\ n+2$ *and* $n+3.$ Then $n\cdot(n+1)\cdot(n+2)\cdot(n+3)+1=n^4+6\cdot n^3+11\cdot n^2+6\cdot n+1=\left(n^2+3\cdot n+1\right)^2.$ This can be verified using the TI-89/VOYAGE 200. See **FIGURES 101-102:**

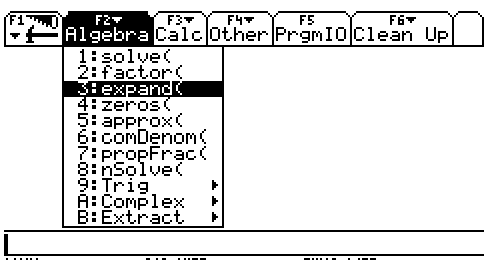

FIGURE 101

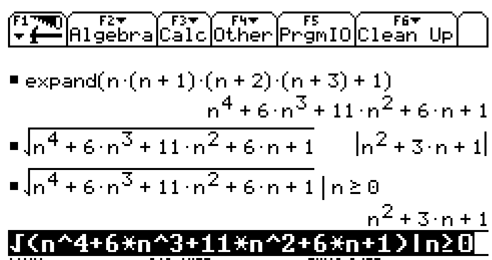

FIGURE 102

(d). The sum of two prime numbers is never a prime number is false! Consider the prime integers 2 and 5. Then $2+5=7$ which is also a prime number. What is true is that the sum of two odd prime numbers is never a prime; for such a sum yields an even integer which is greater than two and hence has two as a divisor and thus is not prime.

(e). The sum of five consecutive integers is divisible by 5 is indeed true; for if we denote the five consecutive integers by $n,\ n+1,\ n+2,\ n+3$ *and* $n+4,$ then $n+(n+1)+(n+2)+(n+3)+(n+4)=5\cdot n+10=5\cdot(n+2).$ Notice that the sum of the five consecutive integers is five times the median which is $n+2.$ One can verify this via our graphing calculator. See **FIGURE 103:**

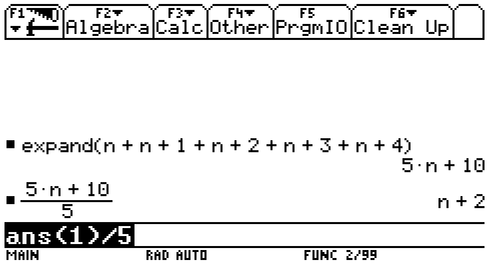

FIGURE 103

(f). If the sides of a right triangle are tripled, then both the perimeter and area of the right triangle are tripled is false. While the perimeter is indeed tripled, the area is increased nine-fold. To cite an example, consider the 3-4-5 right triangle. Note that $3^2+4^2=9+16=25=5^2$. This triangle has a perimeter of 12 $(P=a+b+c=3+4+5=12)$ and an area of 6 $\left(A=\frac{1}{2}\cdot a\cdot b=\frac{1}{2}\cdot 3\cdot 4=\frac{1}{2}\cdot 12=6\right)$. Tripling the sides of the 3-4-5 right triangle yields the similar right triangle 9-12-15. The perimeter is 36 and the area is 54. The new perimeter is three times the original perimeter while the new area is nine times the original area. **FIGURE 104** utilizing the graphing calculator can furnish a formal proof where the area of any right triangle is equal to the product of its legs while the perimeter is the sum of all of its sides. Let the legs be denoted by a *and* b and the hypotenuse by c.

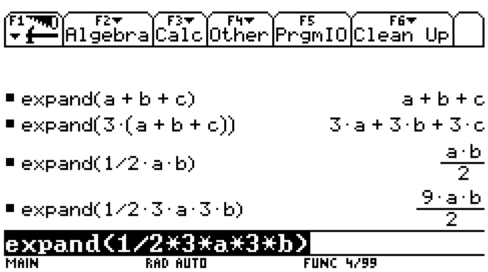

FIGURE 104

THANK YOU FOR YOUR PARTICIPATION AT THIS WORKSHOP DURING THE THE 7TH ANNUAL KCM CONFERENCE ENGAGE 2015 IN LEXINGTON, KY!